Problem A. Generate five matrices A_n of size $n \times n$, $n = m, m+1, \ldots, m+4$, with elements $A_n(i,j) = 1/(i+j-k)$ and five n–vectors e_n with elements $e_n(i) = 1$. Compute the vectors $b_n = A_n e_n$ and find the solution of the five equations $A_n x_n = b_n$. The exact solution is equal to e_n but the computed solution x_n shall only be close to e_n. Compute the relative error $\varepsilon_n = \|x_n - e_n\|_2/\sqrt{n}$ in the computed solution and compare it with the accuracy estimate $\omega_n = 2\mathbf{u}\|A_n\|_2\|A_n^{-1}\|_2$, where $\mathbf{u}$ is the rounding unit.

Task 1: Create the table

ε	ε_m	ε_{m+1}	ε_{m+2}	ε_{m+3}	ε_{m+4}
ω	ω_m	ω_{m+1}	ω_{m+2}	ω_{m+3}	ω_{m+4}

for the prescribed data m and k.

Hint: In MATLAB one has $2\mathbf{u} = \text{eps}$.

Problem B. Create an $m \times n$, $m > n$, random matrix B with the command `rand` and set $A = BB^\top$. Check the rank of A (must be n). Let E_m be the $m \times m$ matrix with elements $E_m(i,j) = 1$ and set $C_p = A + 10^{-p}E_m$.

Task 2: Find a value of p such that $\text{rank}(C_p) = n+1$ but $\text{rank}(C_{p+1}) = n$ for the given data m and n.

Most probably the theoretical rank of C_{p+1} is $n+1$ but MATLAB thinks that the rank is n.

Hint: p is expected to be between 13 and 17.

Task 3: Explain what has happened with the rank determination in the MATLAB environment. Is the inequality $\text{rank}(C_s) > n+1$ possible for some s?

Problem C. For $n = m, m+1, m+2$ create the $n \times n$ Jordan block $J_n(\lambda)$ with eigenvalue λ and set $H_n = I_n - 2e_n e_n^\top/n = I_n - 2E_n/n$, where I_n is the identity $n \times n$ matrix and e_n, E_n are defined in Problems A, B respectively. Set $M_n(\lambda) = H_n J_n(\lambda) H_n$.

Hint: The diagonal and super-diagonal elements of the matrix $J_n(\lambda)$ are $J_n(\lambda)(i,i) = \lambda$, $n = 1, 2, \ldots, n$ and $J_n(\lambda)(i, i+1) = 1$, $n = 1, 2, \ldots, n-1$.

Task 4. Show that the matrix H_n is symmetric and orthogonal.

According to Task 4 the matrix $M_n(\lambda)$ is similar to $J_n(\lambda)$ and therefore has only one eigenvector. However, MATLAB is not aware about that.

Task 5. Try to compute the eigenvalues and eigenvectors of the matrix M_n by the command `eig` for the given data m and λ. Check the condition number and the rank of the computed "eigenvector matrix".

Task 6. Compute the spectral condition numbers of the matrices M_n and explain what happened. What MATLAB says about this problem?

Problem D. Consider the integrals

$$K_m = \int_0^m \frac{\mathrm{d}x}{1+x^m}.$$

Task 7. Compute K_m for the prescribed set of values for m.

Problem E. Consider the double integrals

$$L_k = \int_0^{\pi/2} \int_0^{\pi} \frac{k \mathrm{d}x \mathrm{d}y}{1+x+y+\sin^k(xy)}.$$

Task 8. Compute L_k for the prescribed values of k.

Problem F. Consider the data (x_k, y_k), $k = n+1, n+2, \dots, n+10$, where $x_k = k$ and $y_k = x_k + \sin(x_k)$.

Task 9. Find the spline interpolation of the given data at the points $x_k^0 = k+0.5$, $k = n+1, n+2, \dots, n+9$ for the prescribed value of n.

Task 10. Find the least squares approximation of the given data at the points x_k^0 from Task 9 by a five degree algebraic polynomial.