Continuous beams

A continuous beam is a statically indeterminate multispan beam on hinged support. The end spans may be cantilever, may be freely supported or fixed supported. At least one of the supports of a continuous beam must be able to develop a reaction along the beam axis. An example of a continuous beam is presented in Fig. 1a. The supports are numbered from left to right 1, 2, 3 and 4. The moment of inertia remains constant within the limits of each span, but varies from one span to another.

I. Method of forces.

1. Statically determinate primary system.

The continuous beam in Fig. 1a has three redundant constraints. The statically determinate primary system may be obtained by elimination of constraints considered as redundant. The most intuitive primary system is a simply supported beam, obtained by elimination of internal supports and elimination the constraint developing bending moment in the first fixed support. The most effective (efficient) primary system for continuous beam is proposed by Clapeyron (French engineer and physicist 1799-1864). His primary statically determinate system is obtained by elimination of the constraints which prevent mutual rotation of two neighbouring sections over the supports. With other words the primary system is obtained by putting a hinge at each internal support as shown in Fig. 1b. In our case we should introduce a hinge in the first fixed support (if this point were freely supported the hinges would be introduced over the internal supports only).

2. Canonical equations

The canonical equations expressing mathematically that the angles of every two neighbouring sections over the supports, one with respect to the other remain nil, takes the following form:

\[ \delta_{11} \cdot X_1 + \delta_{12} \cdot X_2 + \delta_{13} \cdot X_3 + \Delta_{1f} = 0, \]

\[ \delta_{21} \cdot X_1 + \delta_{22} \cdot X_2 + \delta_{23} \cdot X_3 + \Delta_{2f} = 0, \]

\[ \delta_{31} \cdot X_1 + \delta_{32} \cdot X_2 + \delta_{33} \cdot X_3 + \Delta_{3f} = 0. \]

The coefficients of all unknowns as well as the free term will be calculated by using the diagrams of the bending moments induced by unit couples acting along the direction of each redundant constraint (Fig. 1c, d and e) and the diagram due to the actual external load (Fig. 1f). In that respect:

\[ \delta_{11} = \sum \int \frac{M_1^2}{EI} ds = \frac{1 \cdot 1 \cdot 3}{3 \cdot EI} = \frac{1}{EI}; \]

\[ \delta_{22} = \sum \int \frac{M_2^2}{EI} ds = \frac{1 \cdot 1 \cdot 3}{3 \cdot EI} + \frac{1 \cdot 1 \cdot 5}{3 \cdot 2EI} = \frac{1.8333}{EI}; \]

\[ \delta_{33} = \sum \int \frac{M_3^2}{EI} ds = \frac{1 \cdot 1 \cdot 5}{3 \cdot 2EI} + \frac{1 \cdot 1 \cdot 6}{3 \cdot EI} = \frac{2.8333}{EI}; \]

\[ \delta_{12} = \delta_{21} = \sum \int \frac{M_1 \cdot M_2}{EI} ds = \frac{1 \cdot 1 \cdot 3}{6 \cdot EI} = 0.5; \]

\[ \delta_{13} = \delta_{31} = \sum \int \frac{M_1 \cdot M_3}{EI} ds = 0; \]
\[ \delta_{23} = \delta_{32} = \sum \int \frac{M_2 \cdot M_3}{EI} \, ds = \frac{1 \cdot 1.5}{6 \cdot 2EI} = 0.4167; \]

\[ \Delta_{1f} = \sum \int \frac{M_1 \cdot M_0}{EI} \, ds = 0; \]

\[ \Delta_{2f} = \sum \int \frac{M_2 \cdot M_0}{EI} \, ds = \frac{1 \cdot 20.5}{3 \cdot 2EI} = \frac{16.667}{EI}; \]

\[ \Delta_{3f} = \sum \int \frac{M_3 \cdot M_0}{EI} \, ds = \frac{1 \cdot 20.5 \cdot 1 \cdot (2.55) \cdot 6}{6 \cdot 6EI} = -101.67 \frac{EI}{EI}. \]

The coefficient \( \delta_{13} \) is zero. It could be concluded that all coefficients to the unknowns in the n-th canonical equation with the exception of the coefficients \( \delta_{n-1,n}, \delta_{n,n} \) and \( \delta_{n,n+1} \) are zero. That simplifies considerably calculation of the basic unknowns \( X_i \) using the force method. This simplification is entirely due to the chosen primary system.

For the continuous beam under consideration the canonical equations take the form:

\[
1 \cdot X_1 + 0.5 \cdot X_2 = 0
\]

\[
0.5 \cdot X_1 + 1.8333 \cdot X_2 + 0.41667 \cdot X_3 + 16.667 = 0
\]

\[
0.41667 \cdot X_2 + 2.8333 \cdot X_3 - 101.667 = 0.
\]

Wherefrom the unknown moments become:

\( X_1 = 10.386; \ X_2 = -20.773; \ X_3 = 38.937. \)

3. Internal forces diagrams

When all the moments of the supports are known, one may proceed with the determination of bending moments within the spans, the shear forces and reactions developed at each support. These computations will be carried out assuming that each span is a simply supported beam and is acted upon both by the applied loads and the moments of the supports just determined (Fig. 1g). The final bending moment diagram \( M_f \), applying the principle of superposition of applied loads, could be obtained as:

\[ M_f = M^{ref} + M^{base}. \]

In the above equation \( M^{ref} \) is the bending moment diagram in the primary system due to the moments of the supports \( X_i \) (Fig. 1h); \( M^{base} \) is the bending moment diagram in the primary system due to the applied loads (Fig. 1i). This diagram coincides completely with \( M^0_f \). The bending moment diagram of the original continuous beam \( M_f \) is shown in Fig. 2a.

Having the bending moment graph \( M_f \) available (Fig. 2a) the shear forces diagram can be derived using the well known expression \( Q_f = dM_f / dx \). This graphics is depicted in Fig. 2b.

Finally, the reaction of any support is equal to the difference between the shear forces acting over two adjacent cross sections located at both sides of the support under consideration. Thus, the numerical value of this reaction will be equal to the jump in the shear diagram over the corresponding support.
Figure 1 Continuous beam. Bending moment diagrams due to the unit couples and applied loads.
4. Verifications

4.1 Equilibrium verification

\[ \sum V = 0 \rightarrow -10.386 + 18.328 - 17.765 + 169.823 - 10 \cdot 6 - 100 = -188.151 + 188.151; \]
\[ \sum M_0 = 0 \rightarrow 10.386 - 18.328 \cdot 3 + 17.765 \cdot 8 - 169.823 \cdot 14 + 20 + 10 \cdot 6 \cdot 11 + 100 \cdot 16 = 2432.51 - 2432.51. \]

If the equilibrium verification is not fulfilled this means that some mistakes could have been made in the bending moment diagrams due to the unit couples of moments or applied loads (in the primary statically determinate system). Another error can be committed in the final bending moment diagram during the summation of the ordinates of \( M_{\text{ref}} \) and \( M_{\text{base}} \).

4.2 Compatibility verification

According to the principle of geometrical compatibility the mutual rotation of points of application of \( X_i \) must be zero, because a constraint between these points is available in the real continuous beam. In that respect the following relations must be satisfied:

\[
\Delta_l = \sum \int \frac{M_1 \cdot M_f}{EI} ds = 0 \rightarrow \frac{1 \cdot (2 \cdot 10.386 - 20.773) \cdot 3}{6 \cdot EI} = 0;
\]
If only one of the compatibility verifications is not fulfilled, say the second equation $\Delta_2 \neq 0$, the probable errors are:

- Error in the derivation of mutual rotation $\delta_{22}$ due to the unit moments $X_2=1$;
- Error in the derivation of mutual rotation $\delta_{2f}$ due to the applied loads;
- The obtained mutual displacements $Z_i$ do not satisfy second canonical equation (this equation does not equal to zero).

II. Slope and deflection method (Displacement method)

1. Kinematically determinate primary system

The unknowns of the slope and deflection method are angles of rotation (rotations) and displacements of the joints of the structure. The total number of unknowns ($n$) is equal to the number of unknown displacements ($n_d$) and rotations of the joints ($n_r$):

$$n = n_d + n_r.$$  

The number of unknown rotations is always equal to the number of the rigid joints of the structure. A joint is considered rigid if at least two members meeting at this joint are rigidly connected one-another.

For the continuous beam under consideration the rigid joints are two (always the number of internal restraints, independently of the type of the end supports). The constraints introduced in order to prevent the rotations of the rigid joints are shown in Fig. 3a.

The number of independent joint deflections ($n_d$) is equal to the degree of freedom of the so-called hinge connected system, obtained by introduction of hinges at all the rigid joints and supports of the original structure (Fig. 3b). The number of joint deflections is always equal to the number of additional bars which should be introduced to make the hinge connected structure geometrically stable. Obviously the hinged system under consideration is statically determinate, compounded by non-singular dyads, or in our case $n_d=0$. Thus, it is not necessary to introduce single bars to prevent joint linear displacements. The kinematically determinate simple system coincides with that in Fig. 3a.

2. Canonical equations

The bending moment diagrams due to the unit rotations of the fixed joints are presented in Figs. 3c and 3d. The bending moment diagram in the kinematically determinate primary system due to the applied loads is shown in Fig. 3e.

The canonical equations of the displacement method are as follow:

$$r_{11} \cdot Z_1 + r_{12} \cdot Z_2 + R_{1f} = 0,$$

$$r_{21} \cdot Z_1 + r_{22} \cdot Z_2 + R_{2f} = 0.$$  

The first equation expresses that in the real structure no reactive moment is developed at the imaginary constraint which prevents the rotation of joint 2. The second expression means that the reaction in the constraint introduced in joint 3, due to rotations of joint 2 and 3 ($Z_1$ and $Z_2$) and due to the applied loads, is equal to zero.
Figure 3 Slope and deflection method for static analysis of the continuous beam
3. Determination of coefficients to the unknowns of the canonical equations and the free terms.

In the above equations \( r_1 \) is the reactive moment due to the rotation of joint 2 through an angle equal to unity; \( r_{12} \) is the reactive moment in the imaginary constraint of joint 2 due to a unit rotation of joint 3; \( r_{22} \) is the reactive moment which arises in the second support when the joint 3 is rotated to unity. The free term \( R_{1f} \) is the reactive moment in the first imaginary constraint due to the applied loads; \( R_{2f} \) is the reaction in the second restraint due to the same loads. \( Z_1 \) and \( Z_2 \) are basic unknowns of the slope and deflection method, namely the rotations of the rigid joints 2 and 3.

The coefficients to the unknowns and the free terms, in the case of continuous beam, should be obtained isolating each of the fixed joints and forming the equilibrium equations of the type \( \Sigma M = 0 \). The coefficients to the unknowns \( r_{ij} \) are obtained in Figs. 3c and 3d next to the bending moment diagrams. The free terms \( R_{if} \) are shown in Fig. 3e.

The canonical system of equations becomes:

\[
2.9333EI \cdot Z_1 + 0.8EI \cdot Z_2 - 20 = 0, \\
0.8EI \cdot Z_1 + 2.1EI \cdot Z_2 + 55 = 0.
\]

Wherefrom the unknown rotations are:

\[
Z_1 = 15.579 / EI; \quad Z_2 = -32.162 / EI.
\]

4. Internal forces diagrams

The final bending moment diagram of the original system can be obtained by the summation of the ordinates to the \( M^0_f \) diagram with those of the unit diagrams being previously multiplied by the magnitude of the unknowns just determined, or:

\[
M_f = M^0_f + M_1 \cdot Z_1 + M_2 \cdot Z_2.
\]

The bending moment diagram is shown in Fig. 3f. This diagram is the same as that previously obtained by the method of forces.

5. Verification

The verification of the slope and deflection method is the static equilibrium. In the case of a continuous beam the equilibrium of moments acting at each of the rigid joints should be checked. If the bending moments at one of the joints do not balance, this means that some mistakes could have been made in computing the value of the corresponding reactions or just the relevant canonical equation is not satisfied.

The equilibrium of the rigid joints is fulfilled, which could be seen in Fig. 3f.
III. Influence lines

Applying the method of forces the internal force at any section of the continuous beam under consideration, or support reaction, could be obtained by the following equation:

\[ S_m^0 = \sum_S S_{m_i} X_i \]

Let us derive the bending moment and shear force influence lines in section \( m \) of the beam, placed at the half of the second span (Fig. 4a) and the support reaction influence line for reaction at joint 2.

![Figure 4 Unit bending moment diagrams](image)

The values of internal forces in section \( m \) at the primary statically determinate system, due to the unit moments \( X_i = 1 \), are as follow (Figs. 4b, c and d):

- \( M_{m,1} = 0 \), \( Q_{m,1} = 0 \);
- \( M_{m,2} = 0.5 \), \( Q_{m,2} = -\frac{1}{5} = -0.2 \);
- \( M_{m,3} = 0.5 \), \( Q_{m,3} = \frac{1}{5} = 0.2 \);

The expressions for determination of corresponding influence lines take the following form:

- \( M_m = M_0^m + M_{m,1} X_1 + M_{m,2} X_2 + M_{m,3} X_3 = M_0^m + 0.5 \cdot X_2 + 0.5 \cdot X_3 \);
- \( Q_m = Q_0^m + Q_{m,1} X_1 + Q_{m,2} X_2 + Q_{m,3} X_3 = Q_0^m - 0.2 \cdot X_2 + 0.2 \cdot X_3 \);
- \( R = R_0^m + R_{1} \cdot X_1 + R_{2} \cdot X_2 + R_{3} \cdot X_3 = R_0^m + \frac{1}{3} \cdot X_1 - 0.53333 R_{2} \cdot X_2 + 0.2 \cdot X_3 \).

1. Internal forces influence lines in statically determinate primary system
The influence lines for internal forces and required reaction in the primary statically determinate system are shown in Fig. 5. These influence lines are derived as the respective lines in the simply supported beams.

Figure 5 Influence lines for the internal forces and the support reaction in the primary system

Now we proceed with influence lines construction for the basic unknowns of the method of forces $X_i$. These unknowns are derived as a solution of the system of canonical equations, which written in a matrix form is:

$$[\delta] \cdot \{X\} = -\{\Delta_f\},$$

wherefrom:

$$\{X\} = -[\delta]^{-1} \cdot \{\Delta_f\} = [\beta] \cdot \{\Delta_f\}.$$

The matrix $[\beta]$ is the inverse matrix of the compliance matrix $[\delta]$ multiplied by -1. Thus, in order to obtain expressions for $X_i$, first we should compute the matrix $[\beta]$.

$$[\beta] = -[\delta]^{-1} = -\left(\frac{1}{EI}\begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1.83333 & 0.41667 \\ 0 & 0.41667 & 2.8333 \end{bmatrix}\right)^{-1} = EI\begin{bmatrix} -1.1642 & 0.3285 & -0.048309 \\ 0.3285 & -0.657 & 0.096618 \\ -0.048309 & 0.096618 & -0.36715 \end{bmatrix}$$

2. Influence lines for $\Delta_f$

Next, we should construct the influence lines for the mutual rotations $\Delta_f$ in the simple statically determinate system. Influence lines for displacements can be derived as elastic curve of the road lane caused by a unit load (unit couple of moments) along the direction of required displacement. The elastic curve of the road lane can be obtained as a bending moment diagram in a fictitious conjugate beam loaded with the corresponding fictitious distributed and concentrated loads.
2. 1 Influence line for $\Delta_{1f}$

![Diagram showing influence lines for $\Delta_{1f}$, $\Delta_{2f}$, and $\Delta_{3f}$.]

Figure 6 Influence lines for $\Delta_{if}$
In order to derive \( \Delta_1 f \) we should first introduce a unit load (unit bending moment) in the direction of required rotation and construct the corresponding bending moment diagram. This diagram is \( M_1 \) from Fig. 4b.

Next, the conjugate beam must be formed. Starting from left to right, along the beam axis, a mutual rotation is possible between the ground and the left end of the beam at joint 1. Therefore, the fictitious concentrated force must be introduced at the corresponding section of the conjugate beam. This force is denoted \( \varphi_1 \), in accordance with its physical meaning, and is shown in Fig. 6b.

Likewise mutual rotations are allowed between every both adjacent sections of the hinges at joints 2 and 3. The fictitious concentrated forces corresponding to these rotations are \( \Delta \varphi_2 \) and \( \Delta \varphi_3 \).

Considering the end of the beam mutual rotation and mutual vertical displacement is allowed between the end section of the beam and the ground. These concentrated fictitious moment and force must be introduced at the end of the beam respectively \( \varphi_4 \) and \( w_4 \).

Sections of the road lane for which the vertical displacements are zero could be replaced by hinges in the conjugate beam. These sections are joints 1, 2, 3 and 4, because there are vertical supports at these points in the simple system. Thus, in the conjugate beam we can introduce hinges at sections corresponding to these supports (Fig. 6b).

Finally, the conjugate beam must be loaded with the distributed transverse load \( q^{\text{fict}} = \frac{M_1}{(EI \cdot \cos \alpha)} \) (Fig. 6b). In order to avoid the calculation of all fictitious concentrated loads the conjugate beam could be supported as statically determinate system, as shown in Fig. 6c.

The bending moment diagram of conjugate beam, which coincides with required influence line \( "\Delta_1 f" \) is shown in Fig. 6d.

**2. 2 Influence line for \( \Delta_2 f \)**

In order to obtain \( \Delta_2 f \) influence line we should introduce a unit couple of moments at joint 2 and trace the bending moment diagram in the simple system. This is the bending moment diagram \( M_2 \) given in Fig. 4c.

The conjugate beam for determination of \( \Delta_2 f \) is the same as the one constructed for \( \Delta_1 f \), given in Figs. 6c and 6e. The distributed transverse load here is \( q^{\text{fict}} = \frac{M_2}{(EI \cdot \cos \alpha)} \) and this load is presented in Fig. 6e. The influence line \( "\Delta_2 f" \) is obtained in Fig. 6f.

**2. 3 Influence line for \( \Delta_3 f \)**

The bending moment diagram corresponding to mutual rotation at joint 3 is \( M_3 \) given in Fig. 4d. The conjugate beam with the relevant loads is shown in Fig. 6g. The fictitious bending moment diagram presenting the required influence line \( "\Delta_3 f" \) is depicted in Fig. 6h.

**3. Influence lines for the basic unknowns of the force method \( X_i \)**

Influence lines for bending moments at the supports of the continuous beam can be derived by the equation \( \{ X \} = [ \beta ] \cdot \{ \Delta_f \} \). The same expressions in expanded form are:
\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} = EI \begin{bmatrix}
-1.1642 & 0.3285 & -0.048309 \\
0.3285 & -0.657 & 0.096618 \\
-0.048309 & 0.096618 & -0.36715
\end{bmatrix} \begin{bmatrix}
\Delta_{1f} \\
\Delta_{2f} \\
\Delta_{3f}
\end{bmatrix},
\]

or:
\[
X_1 = -1.1642 \cdot EI \cdot \Delta_{1f} + 0.3285 \cdot EI \cdot \Delta_{2f} - 0.048309 \cdot EI \cdot \Delta_{3f},
\]
\[
X_2 = 0.3285 \cdot EI \cdot \Delta_{1f} - 0.657 \cdot EI \cdot \Delta_{2f} + 0.096618 \cdot EI \cdot \Delta_{3f},
\]
\[
X_3 = -0.048309 \cdot EI \cdot \Delta_{1f} + 0.096618 \cdot EI \cdot \Delta_{2f} - 0.36715 \cdot EI \cdot \Delta_{3f}.
\]

The influence lines \(X_i\) are obtained by the summation of the ordinates of the "\\(\Delta_{ij}\\)" influence lines previously multiplied by the coefficients of matrix \([\beta]\) in accordance with the above expressions. These lines are drawn in Fig. 7.

4. Influence lines for the internal forces and support reaction of the original indeterminate beam

Having influence lines for unknown moments \(X_i\) and internal forces in the statically determinate system available, we can obtain required internal forces influence lines in the original continuous beam. This will be done by the summation of relevant ordinates of the influence lines according to the following expressions:

\[
M_m = M_0^0 + 0.5 \cdot X_2 + 0.5 \cdot X_3;
\]

\[
Q_m = Q_0^0 - 0.2 \cdot X_2 + 0.2 \cdot X_3;
\]

\[
R = R_0^0 + \frac{1}{3} \cdot X_1 - 0.53333R_2 \cdot X_2 + 0.2 \cdot X_3.
\]

The corresponding influence lines for the internal forces in section \(m\) and the support reaction at joint 2 are given in Fig. 8.
Figure 7 Influence lines for bending moments over the supports $X_i$
Figure 8 Influence lines for internal forces and support reaction in the continuous beam
5. Verification

In order to verify the obtained influence lines for the given indeterminate continuous beam we shall derive the bending moments over the supports ($X_i$), internal forces at section $m$ ($M_m$ and $Q_m$) and support reaction at joint 2 ($R$) by using their influence lines and we shall compare the results with those previously obtained.

\[
\begin{align*}
X_1 &= 20 \left[ \frac{1}{6 \cdot 1.25} \left( -11 \cdot 0 + 18 \cdot 0.20097 - 9 \cdot 0.21890 + 2 \cdot 0.12738 \right) - \right. \\
&\quad -10 \frac{2}{3} \left( \frac{0}{2} + 2 \cdot 0.09511 + 0.10870 + 2 \cdot 0.06793 + \frac{0}{2} \right) + 100 \cdot 0.09662 = 10.386; \\
X_2 &= -20 \left[ \frac{1}{6 \cdot 1.25} \left( -11 \cdot 0 + 18 \cdot 0.40195 - 9 \cdot 0.43780 + 2 \cdot 0.25476 \right) + \\
&\quad +10 \frac{2}{3} \left( \frac{0}{2} + 2 \cdot 0.19022 + 0.21739 + 2 \cdot 0.13587 + \frac{0}{2} \right) - 100 \cdot 0.19324 = -20.773; \\
X_3 &= -20 \left[ \frac{1}{6 \cdot 1.25} \left( -11 \cdot 0 + 18 \cdot 0.11322 - 9 \cdot 0.21135 + 2 \cdot 0.20380 \right) - \\
&\quad -10 \frac{2}{3} \left( \frac{0}{2} + 2 \cdot 0.72283 + 0.82609 + 2 \cdot 0.51630 + \frac{0}{2} \right) + 100 \cdot 0.73430 = 38.937. \\
M_m &= 20 \cdot 0.21014 - 10 \frac{2}{3} \left( \frac{0}{2} + 2 \cdot 0.26630 + 0.30435 + 2 \cdot 0.19022 + \frac{0}{2} \right) + 100 \cdot 0.27053 = \\
&= 19.082; \\
Q_m &= -20 \cdot 0.11305 - 10 \frac{2}{3} \left( \frac{0}{2} + 2 \cdot 0.18261 + 0.2087 + 2 \cdot 0.13043 + \frac{0}{2} \right) + 100 \cdot 0.18551 = \\
&= 7.9421; \\
R &= 20 \left[ \frac{1}{6 \cdot 1.25} \left( -11 \cdot 1 + 18 \cdot 1.00872 - 9 \cdot 0.76419 + 2 \cdot 0.38757 \right) - \\
&\quad -10 \frac{2}{3} \left( \frac{0}{2} + 2 \cdot 0.27772 + 0.31739 + 2 \cdot 0.19837 + \frac{0}{2} \right) + 100 \cdot 0.28213 = 18.329. \\
\end{align*}
\]
IV. Appendix

Some expressions for numerical integration and differentiation are given in this section. These expressions are applicable for functions with known numerical values in equal subintervals.

1. Numerical integration

The expressions presented below are valid for smooth functions in the interval into consideration.

When a square or cubic parabola is defined over three ordinates (respectively the interval is divided into two equal subintervals), the area of the obtained figure (shaded area) is:

\[ A = \frac{2\lambda}{3} \left( \frac{a + 2b + c}{2} \right) \]  

(1)

When a cubic parabola is defined over four ordinates (respectively the interval is divided into three equal subintervals), the area of the generated figure (shaded area) is:

\[ A = \frac{3\lambda}{8} \left( a + 3b + 3c + d \right) \]  

(2)

By multiple application of equation (1), an expression valid for arbitrary even number of subintervals can be derived. In case of six subintervals for example the area of the figure becomes:

\[ A = \frac{2\lambda}{3} \left( \frac{a + 2b + c + 2d + e + 2f + g}{2} \right) \]  

(3)
By multiple application of equation (2), an expression valid for number of subintervals divisible
by three can be derived (n/3 is a whole number /integer/, where n is the number of subintervals).
In case of six subintervals for example the area of the figure becomes:

\[
A = \frac{3\lambda}{8} (a + 3 \cdot b + 3 \cdot c + 2 \cdot d + 3 \cdot e + 3 \cdot f + g) \quad (4)
\]

2. Numerical differentiation

When a square parabola is defined over three ordinates (the interval is divided into two equal
subintervals), the first derivative at point with ordinate a (the slope of the tangent line at this
point) could be obtained by the following expression:

\[
\tan \varphi_a = \frac{1}{6 \cdot \lambda} (-11 \cdot a + 18 \cdot b - 9 \cdot c + 2 \cdot d) \quad (7)
\]

The first derivative at point with ordinate b is:

\[
\tan \varphi_b = \frac{1}{6 \cdot \lambda} (a - c) \quad (8)
\]
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